An overview of lattice reduction algorithms

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ENS de Lyon

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Two main uses of lattice reduction in crypto

Context 1: cryptanalysis of lattice-based cryptosystems.

- NTRU (encryption & signature)
- All cryptosystems based on SIS and LWE and their ring variants

$\Rightarrow$ Huge dims, small entries, very expensive task

Context 2: cryptanalysis of other cryptosystems.

- Variants of RSA (with Coppersmith's method)

$\Rightarrow$ Large dims, huge entries, quite cheap

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Roadmap

Goal of this talk

An introductive overview on lattice reduction algorithms

1. **Background on lattices**
2. The lattice reduction framework
3. Strong but slow: BKZ
4. Solving the SIS problem
5. Weak but fast: LLL

Euclidean lattices

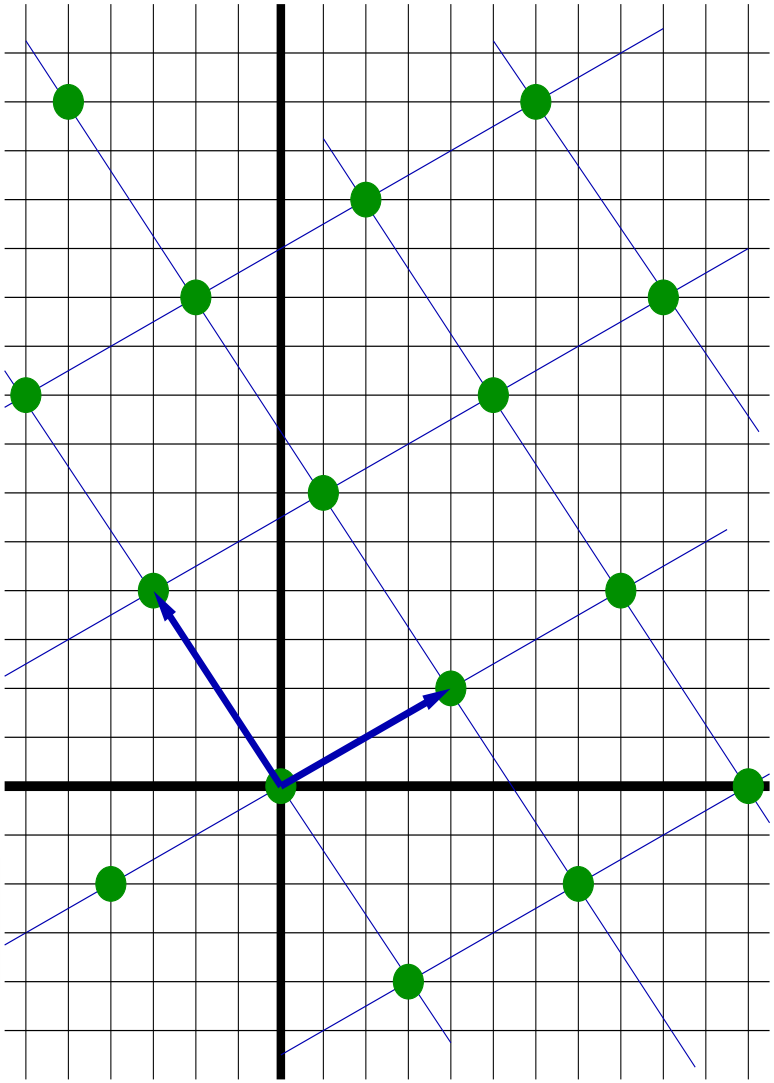
Lattice $\equiv \left\{\sum_{i\leq n} x_i \mathbf{b}_i : x_i \in \mathbb{Z}\right\}$,
for linearly indep. $\mathbf{b}_i$'s in $\mathbb{R}^n$,
referred to as **lattice basis**

Bases are **not unique**, but can be obtained from one another by integer transforms of determinant ± 1:

$$\begin{bmatrix} -2 & 1 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Lattice reduction

Find a short basis, given an arbitrary one



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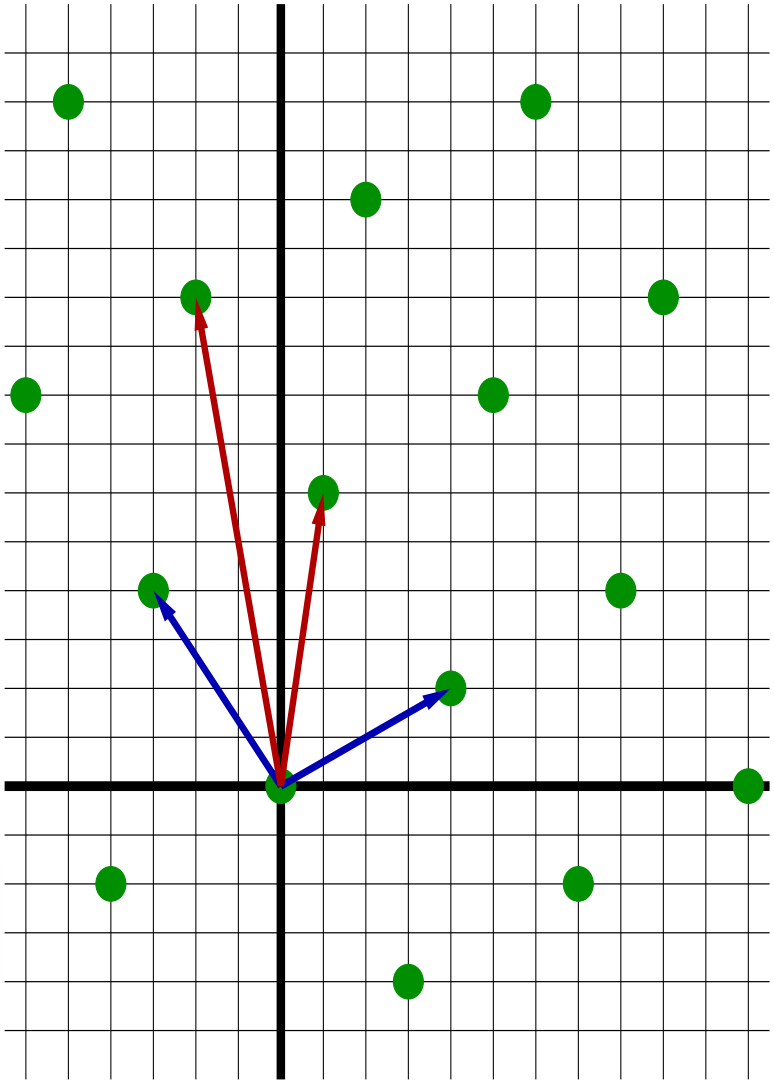

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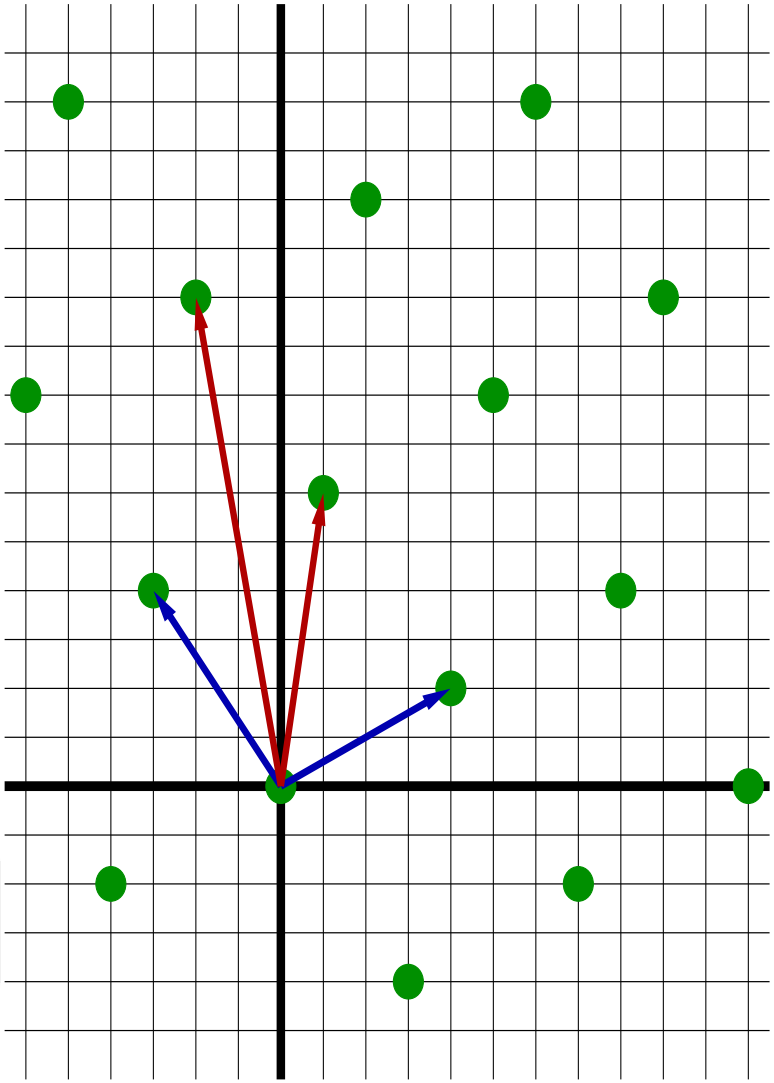

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Lattice invariants

Minimum:
$\lambda(L) = \min(\|\mathbf{b}\| : \mathbf{b} \in L \setminus \mathbf{0})$

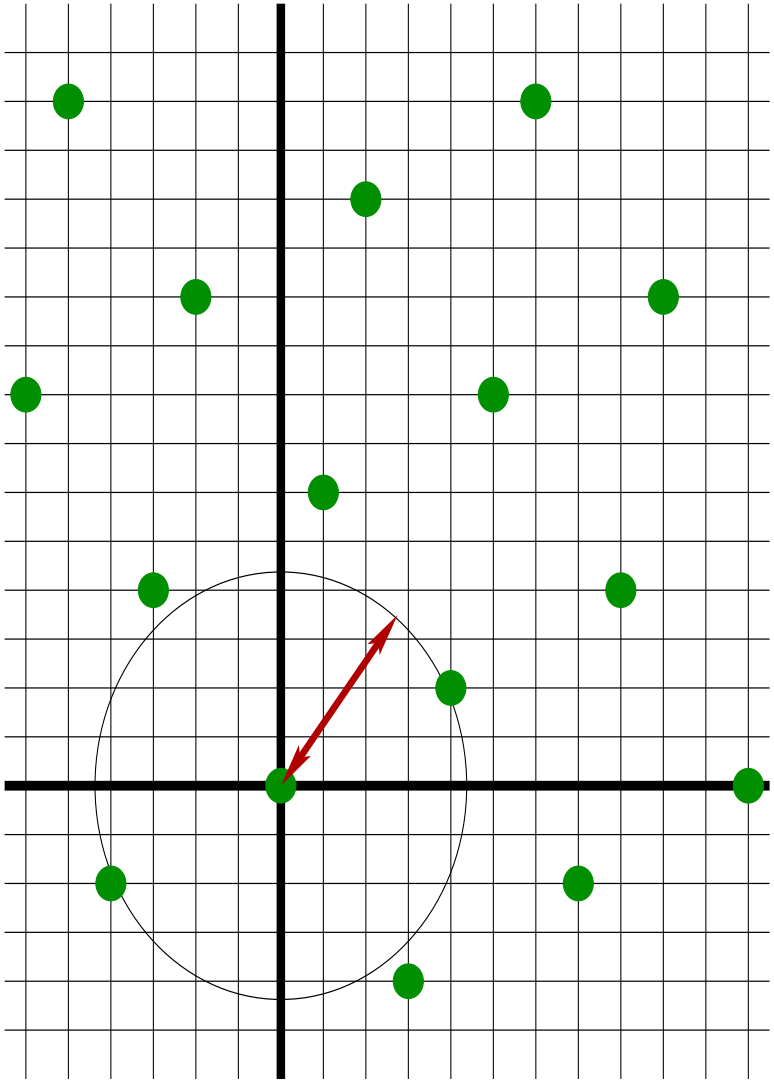
Determinant:
$\det L = |\det(\mathbf{b}_i)_i|$, for any basis

Minkowski theorem

$\lambda(L) \leq \sqrt{n} \cdot (\det L)^{\frac{1}{n}}$, for any L of dim n

Lattice reduction

Find a basis that is short compared to $\lambda(L)$ and/or $(\det L)^{\frac{1}{n}}$



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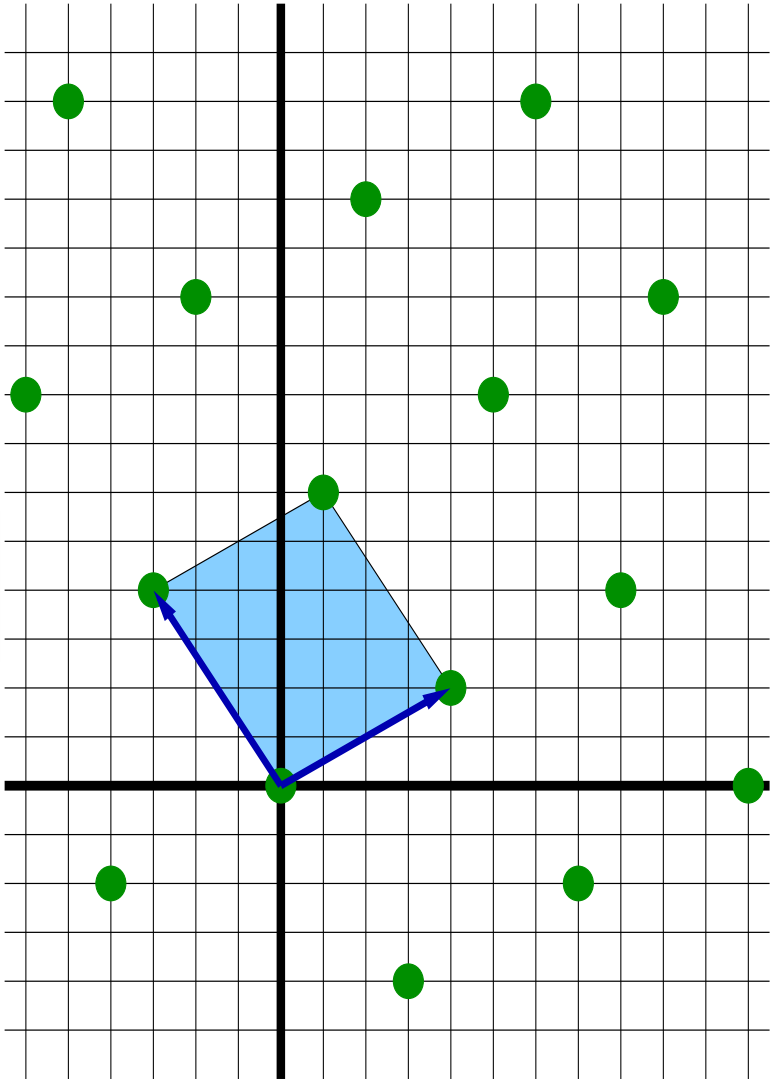

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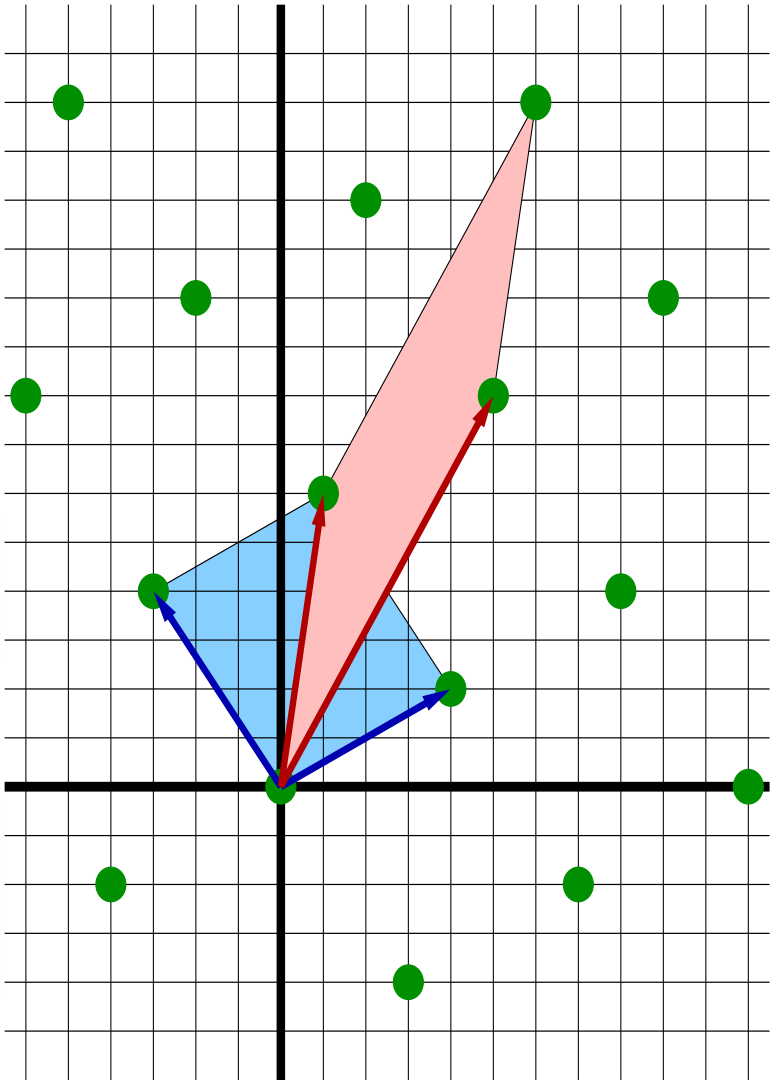

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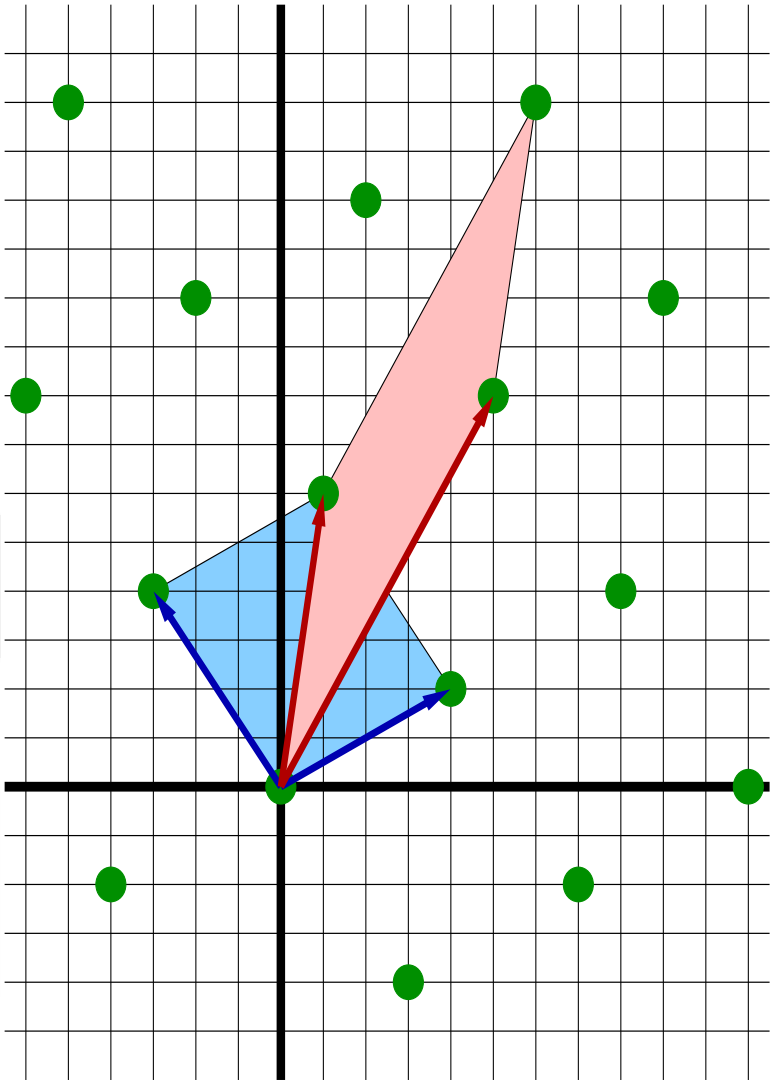

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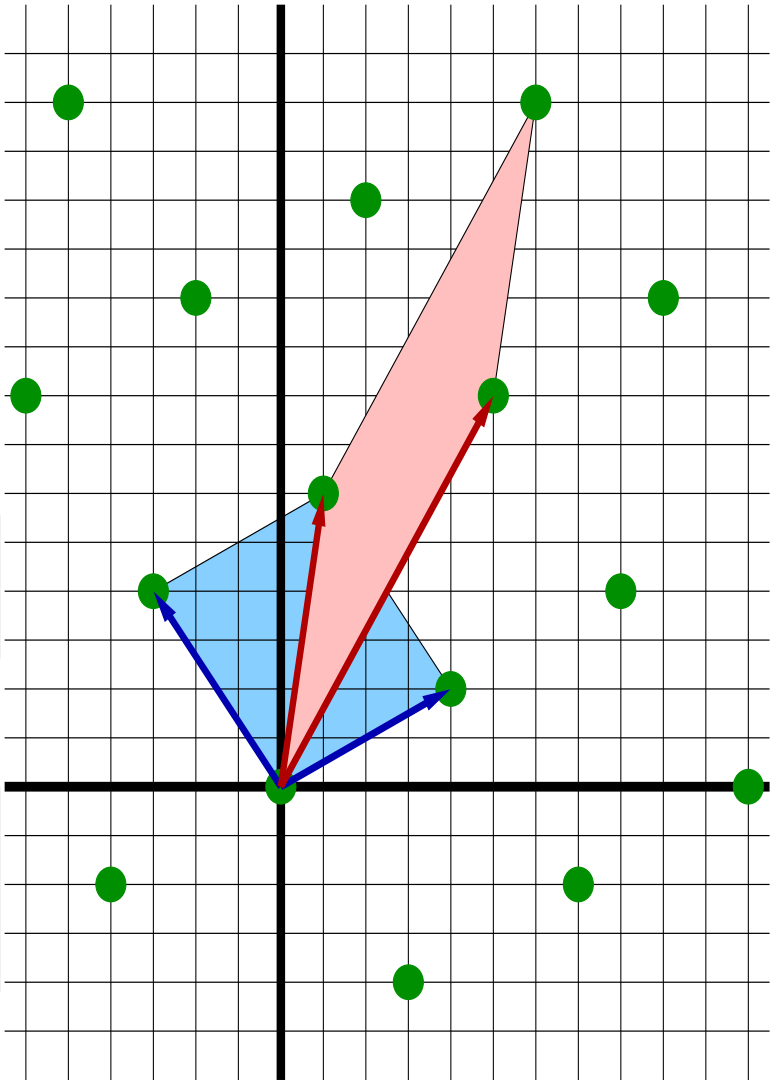

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Computational problems on lattices

The Shortest Vector Problem: SVP_γ

Given a basis of L, find $\mathbf{b} \in L$ s.t.: $0 < \|\mathbf{b}\| \leq \gamma \cdot \lambda(L)$

Effective Minkowski theorem: HSVP_γ

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Many other problems: BDD_γ, CVP_γ, uSVP_γ, SIVP_γ, etc.

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The bad news...

- SVP_γ is NP-hard for $\gamma = O(1)$ (under random. red.) [Ajt98]
- SVP_γ, HSVP_γ, BDD_γ... in P for $\gamma = 2^{\Omega\left(n\frac{\log\log n}{\log n}\right)}$ [Sch87,MiVo10]

When $\gamma \geq n^{\Omega(1)}$, the cost of the best known algorithm is:

$$\mathcal{P}oly(m, \log \|B\|) \cdot \left(1 + \frac{n}{\log \gamma}\right)^{O\left(1 + \frac{n}{\log \gamma}\right)}.$$

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From n^2 to $n(n+1)/2$ variables: QR-factorisation

Goal of lattice reduction

Given $B \in \mathbb{R}^{n \times n}$ full-rank, find $U \in \mathrm{GL}_n(\mathbb{Z})$ s.t. $B \cdot U$ has small coeffs, i.e., its columns have small euclidean norms

As $\|\cdot\|$ is invariant under rotations, we may work on triangular matrices:

$$B = \underbrace{Q}_{\text{orthogonal}} \cdot \underbrace{R}_{\text{up-triangular}} \qquad \text{(QR-factorisation)}$$

- r_{11} is the norm of $\mathbf{b}_1$
- r_{ii} is the norm of the proj. of $\mathbf{b}_i$ orthogonally to $(\mathbf{b}_j)_{j<i}$
- This is equivalent to Gram-Schmidt Orthogonalisation

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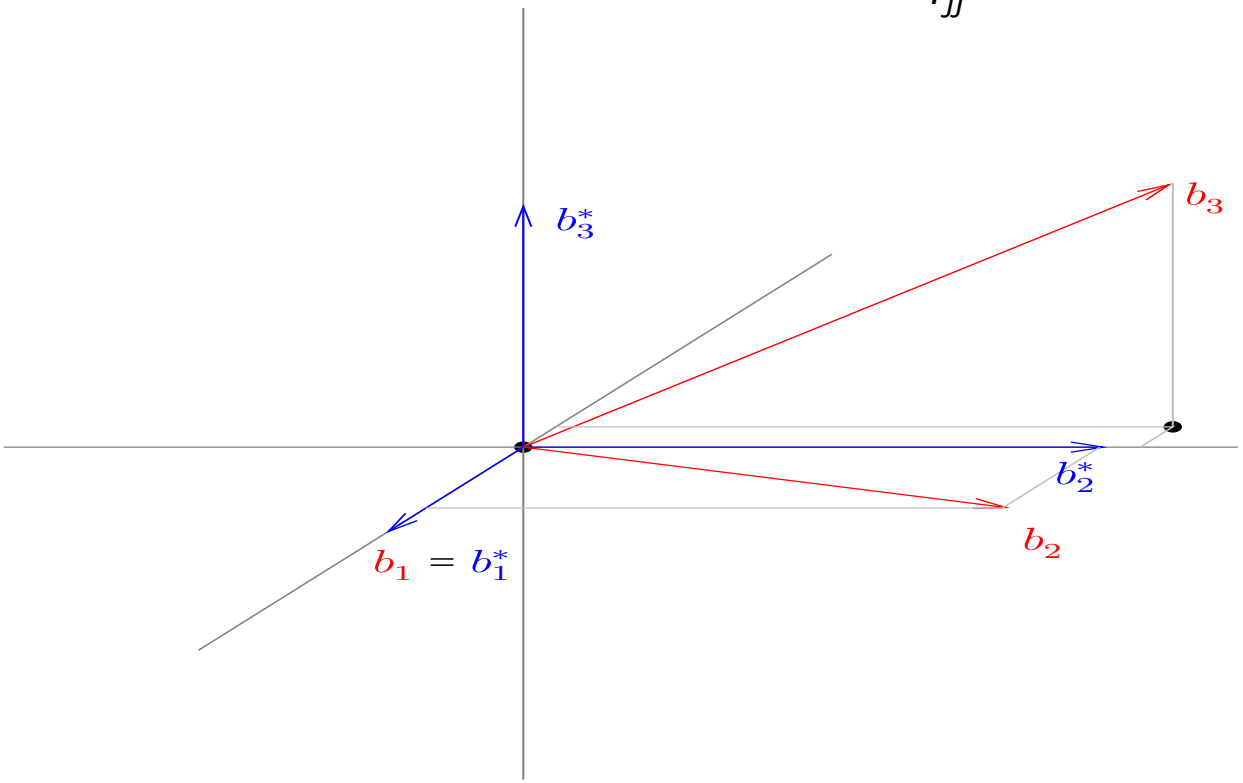
QR-factorisation and GSO

QR is equivalent to Gram-Schmidt Orthogonalisation:

- For all i, $\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j<i} \mu_{ij} \mathbf{b}_j^*$ is the projection of $\mathbf{b}_i$ orthogonally to $\mathrm{Span}_{\mathbb{R}}(\mathbf{b}_1, \ldots, \mathbf{b}_{i-1})$.
- We have $\|\mathbf{b}_i^*\| = r_{ii}$ and $\mu_{ij} = \frac{r_{ji}}{r_{jj}}$ for $i > j$.



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But other coefficients of R may have changed:
If they were small, they may not be small anymore...

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This modifies the top of the j-th column.
$\Rightarrow$ Proceed from bottom to top.

Size-reduced basis

A basis is said size-reduced if $|r_{ij}| \leq r_{ii}/2$ for all $i < j$

- Size-reduction grants control of the off-diagonal coeffs
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Where are we now?

Goal of lattice reduction

Given $R \in \mathbb{R}^{n \times n}$ up-triangular, find $U \in \mathrm{GL}_n(\mathbb{Z})$
s.t. the R-factor of $R \cdot U$ has small diagonal coeffs

What does it mean, as the product of the r_{ii}'s is constant?
We want to

- make the first r_{ii}'s small,
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The best we can do: HKZ

HKZ-reduction

R up-triangular is HKZ-reduced if

- $r_{11} = \lambda(L)$ with $L = \sum_i \mathbb{Z}\mathbf{r}_i$
- and $(r_{ij})_{i,j>1}$ is HKZ-reduced

In the worst case, we have, for all $i \leq n$:

$$r_{ii} \approx \sqrt{n-i+1} \cdot \left(\prod_{j=i}^{n} r_{jj} \right)^{\frac{1}{n-i+1}}$$

Fixing r_{nn} fixes the other r_{ii}'s.
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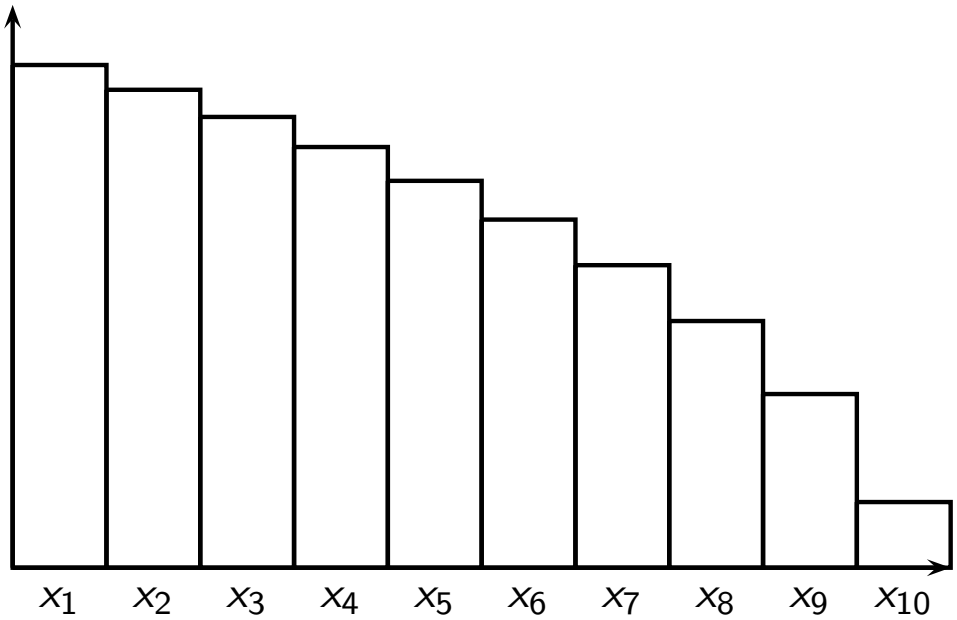
Worst-case HKZ profile:

$$\begin{aligned} x_i &= \log r_{ii} \\ &= O(\log^2(n-i+1)) \end{aligned}$$

Cost of HKZ:

- Computationally eq. to SVP
- Time & space $2^{O(n)}$



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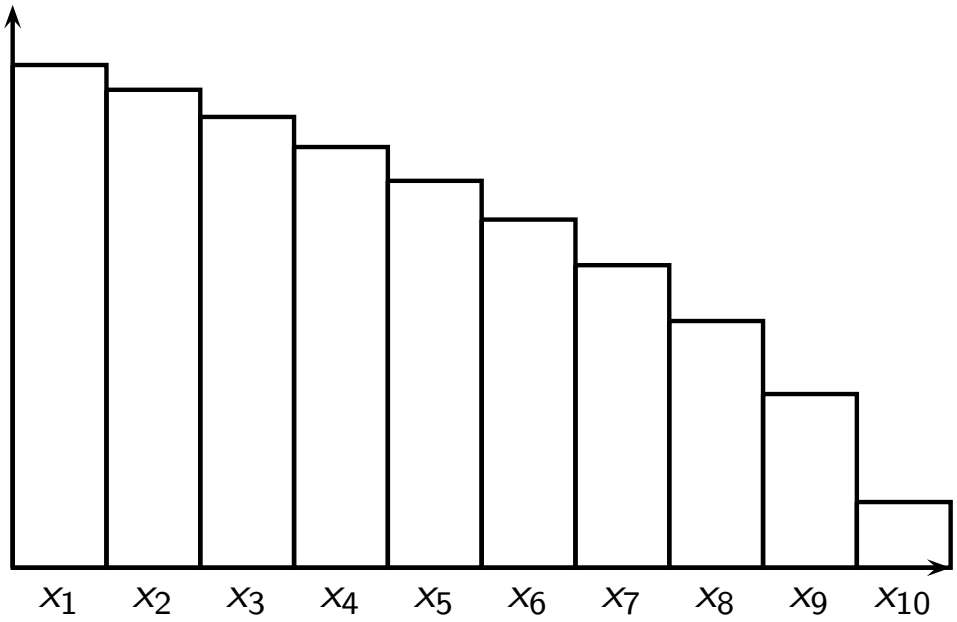

Lattice reduction: the rules of the game

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Given $R \in \mathbb{R}^{n \times n}$ up-triangular, find $U \in \mathrm{GL}_n(\mathbb{Z})$
s.t. the R-factor of $R \cdot U$ has small diagonal coeffs

HKZ too costly... What can we do?

1. Swap two consecutive vectors s.t. $r_{i+1,i+1} \ll r_{i,i}$ [LLL82]
2. Balance the diag. coeffs locally by applying lattice reduction (e.g., HKZ) to a diag. submatrix of R [Sch87]

$$\begin{bmatrix} \ddots & & & \vdots \\ & \begin{bmatrix} r_{ii} & \cdots & \cdots \\ & \ddots & \vdots \\ & & r_{jj} \end{bmatrix} & & \vdots \\ & & \ddots & \\ & \cdots & & \end{bmatrix} \cdot \begin{bmatrix} Id & \cdots & & \vdots \\ & \begin{bmatrix} & & \\ & U_{HKZ} & \\ & & \end{bmatrix} & & \vdots \\ & & \ddots & \\ & \cdots & & Id \end{bmatrix}$$

Lattice reduction: the rules of the game

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Given $R \in \mathbb{R}^{n \times n}$ up-triangular, find $U \in \mathrm{GL}_n(\mathbb{Z})$
s.t. the R-factor of $R \cdot U$ has small diagonal coeffs

HKZ too costly... What can we do?

1. Swap two consecutive vectors s.t. $r_{i+1,i+1} \ll r_{i,i}$ [LLL82]
2. Balance the diag. coeffs locally by applying lattice reduction (e.g., HKZ) to a diag. submatrix of R [Sch87]

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1. Background on lattices
2. The lattice reduction framework
3. **Strong but slow: BKZ**
4. Solving the SIS problem
5. Weak but fast: LLL

BKZ: A global reduction approach

[Sch87,ScEu91]: Do HKZ on k-dim. diagonal submatrices of R

BKZ$_k$, simplified version

Input: $R \in \mathbb{R}^{n \times n}$ up-triangular
Repeat ... times
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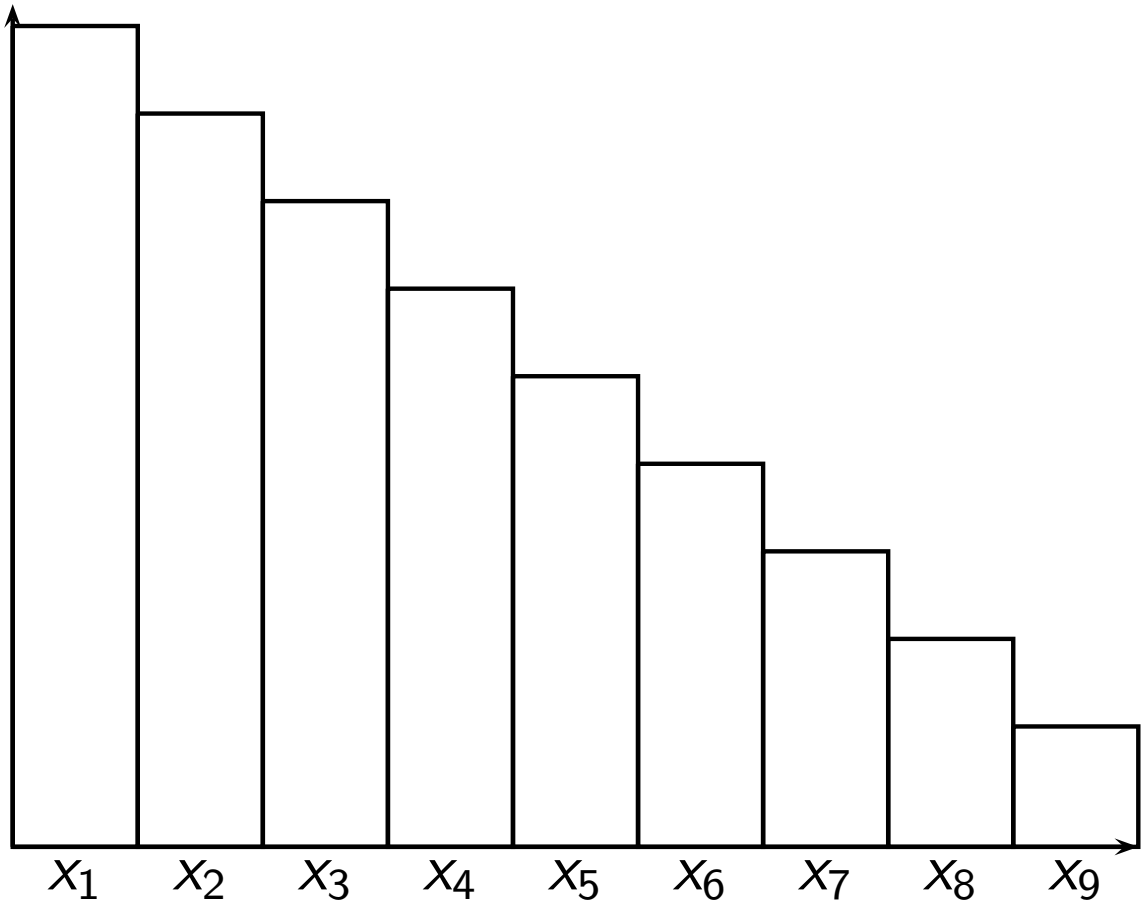
Regularity assumption: Each HKZ-reduction gives a worst-case profile



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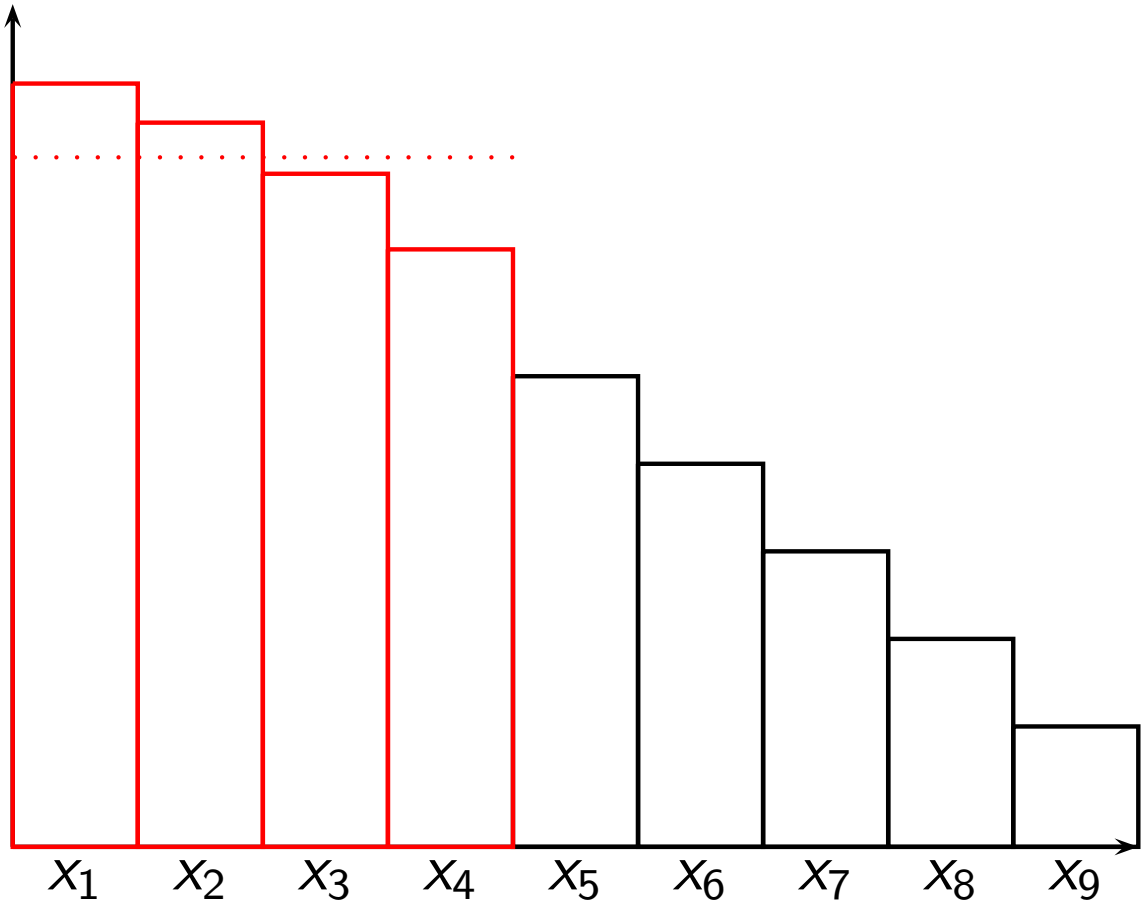

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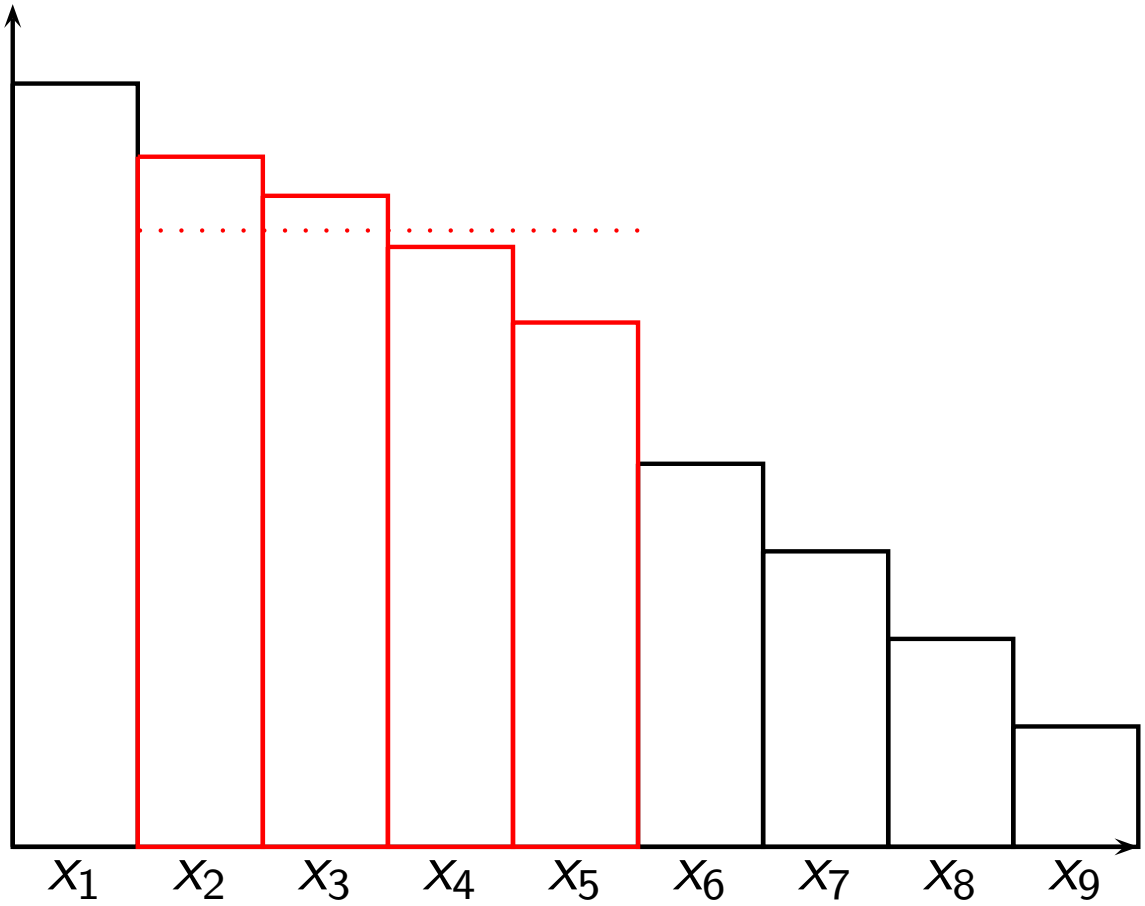

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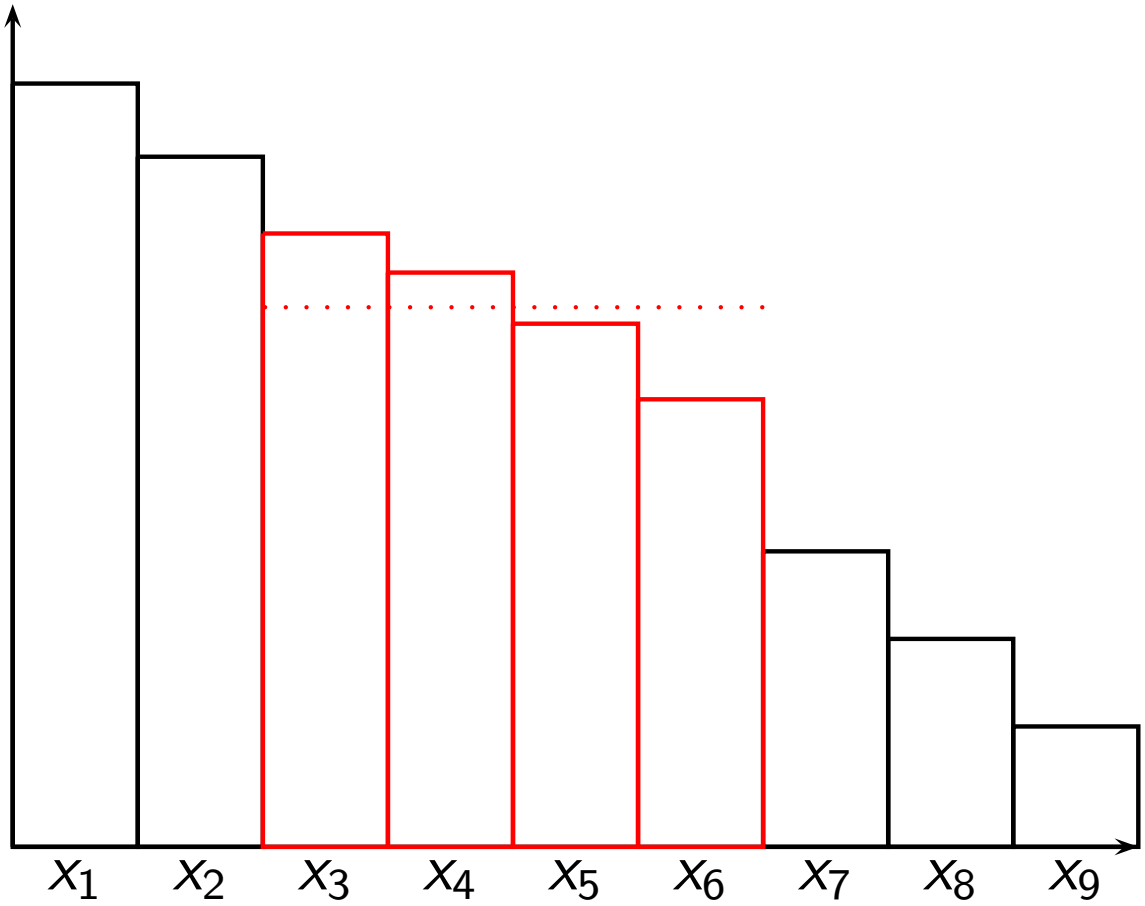

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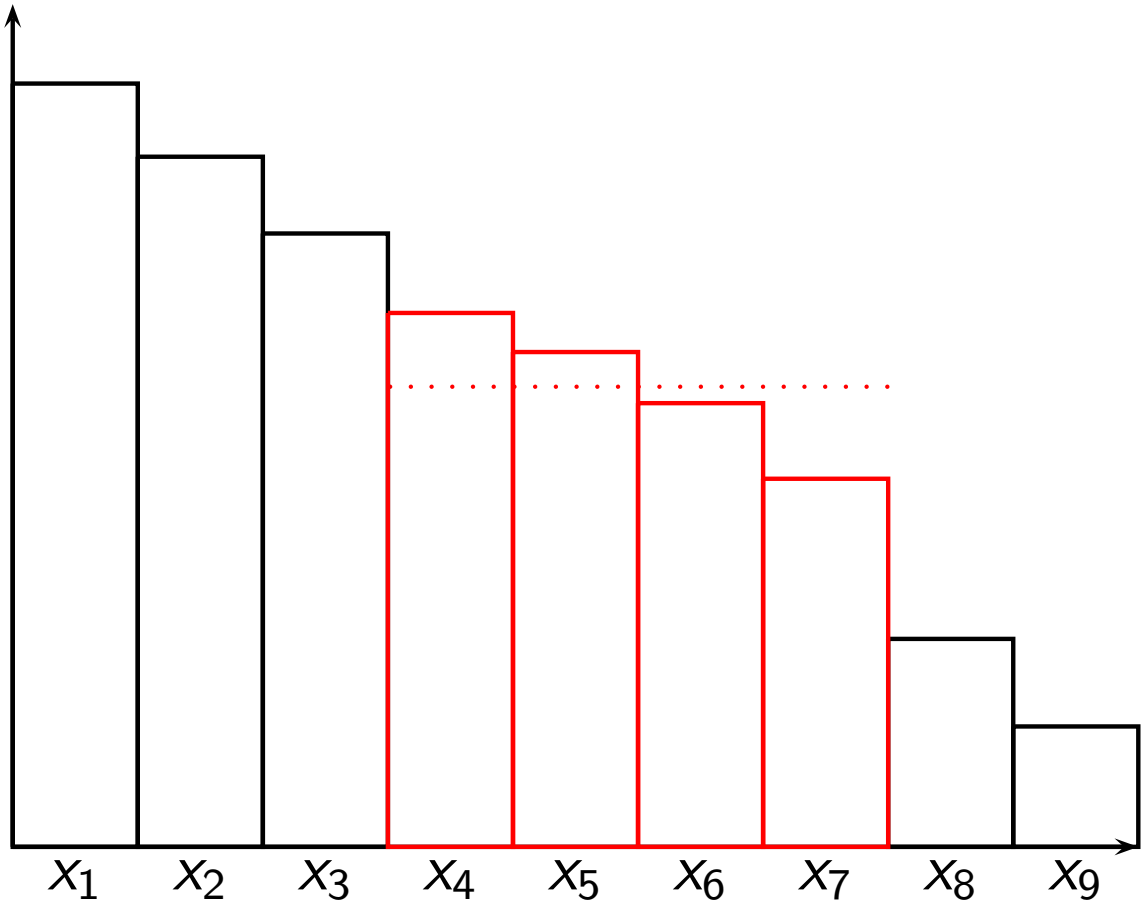

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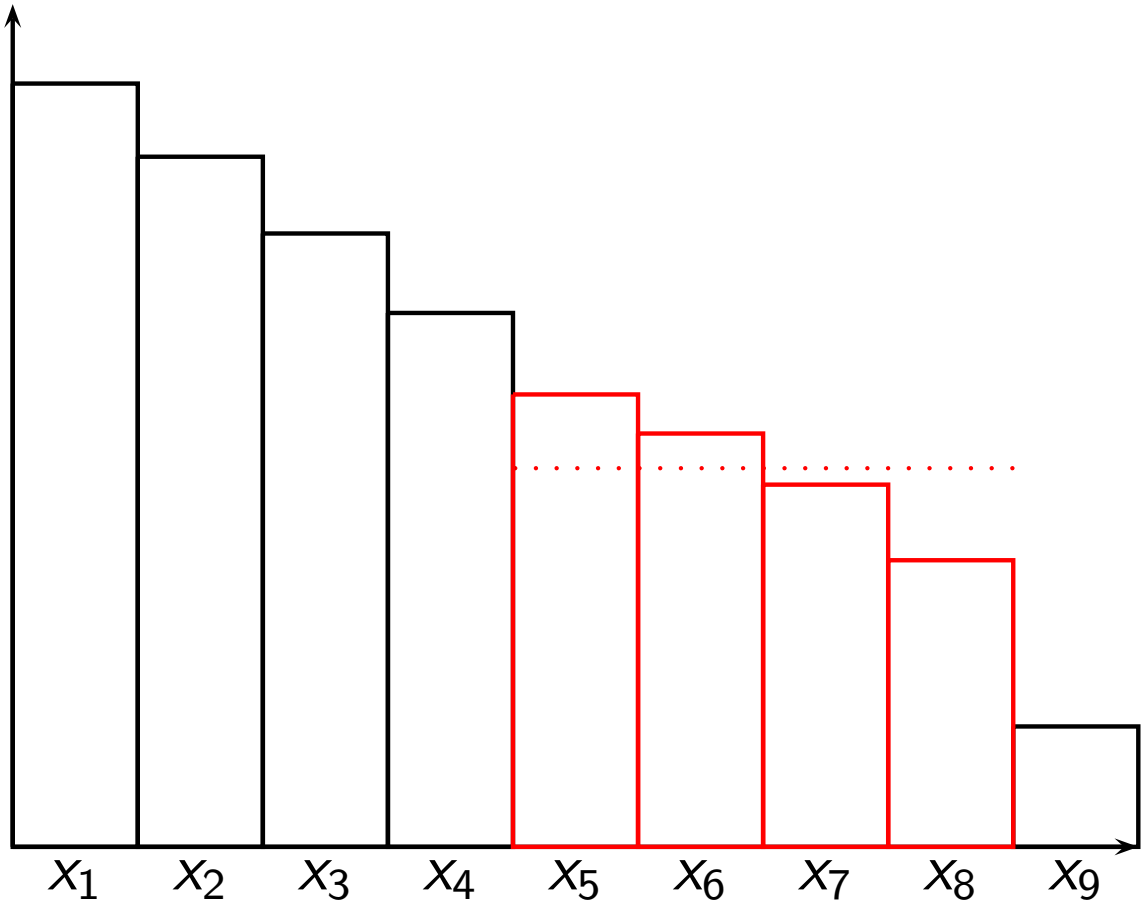

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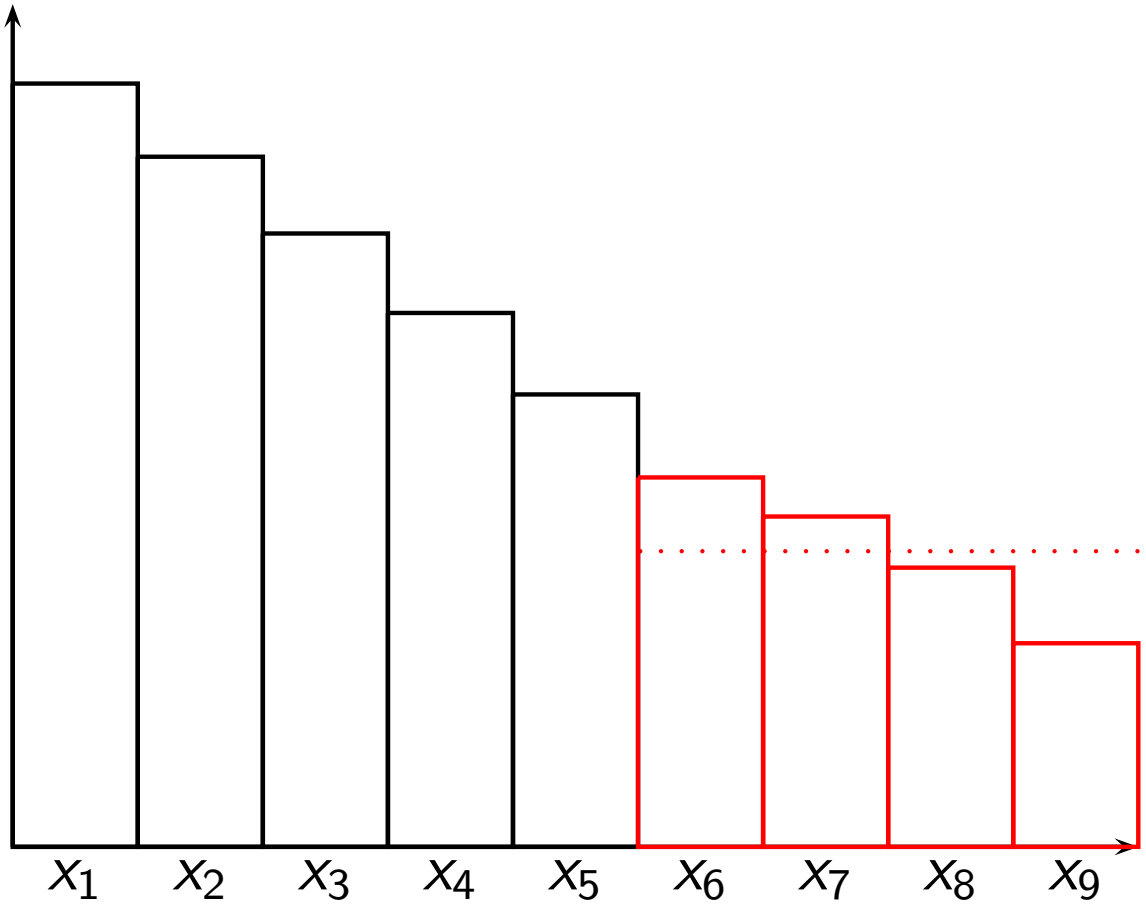

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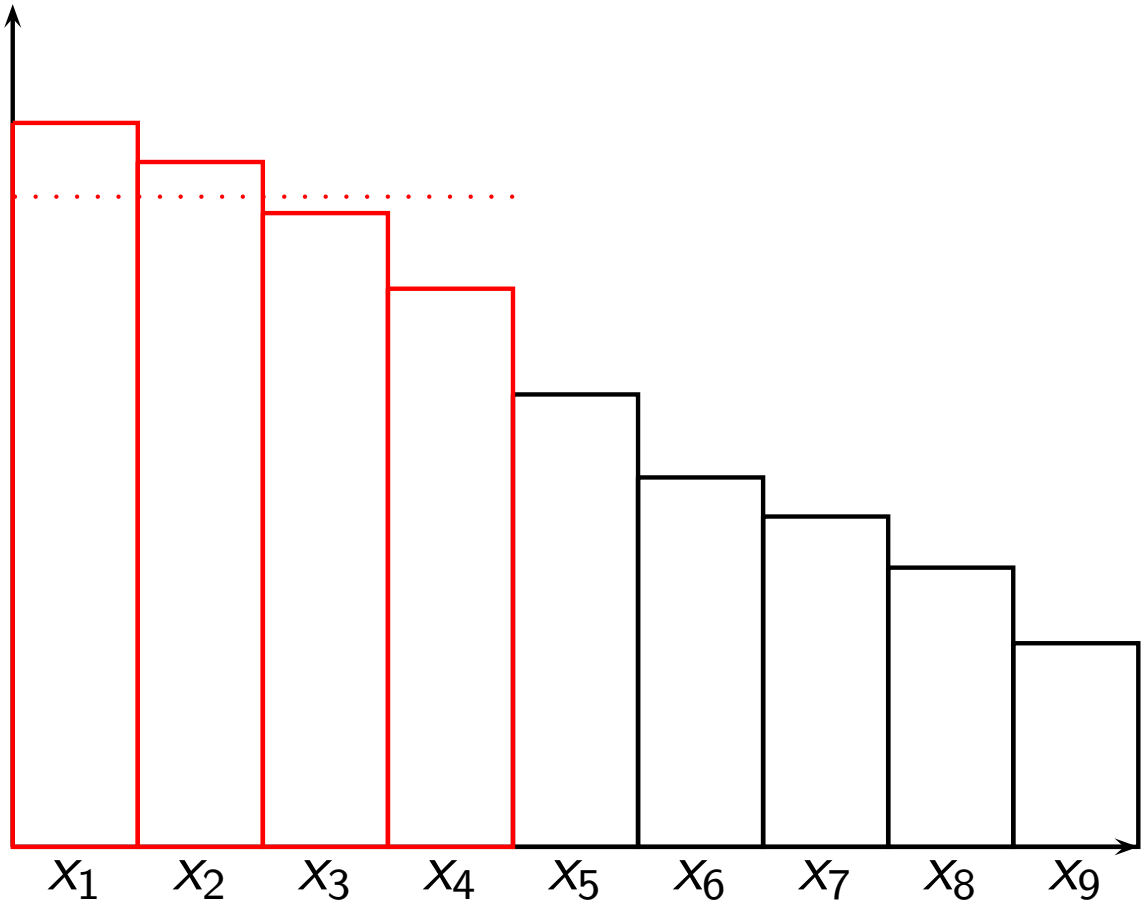
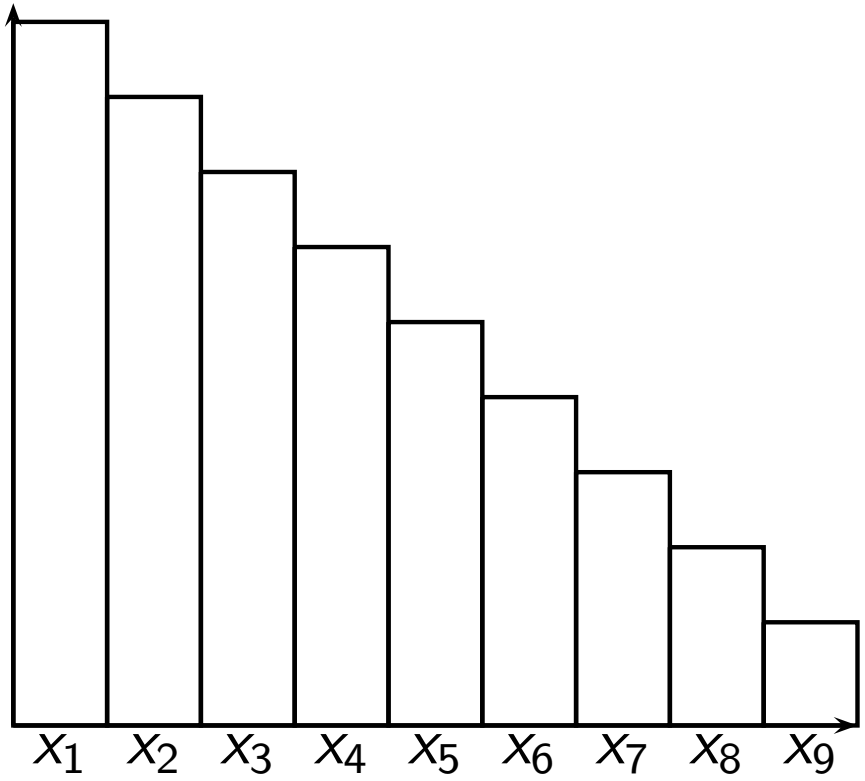


BKZ_k's sandpile as a dynamic system



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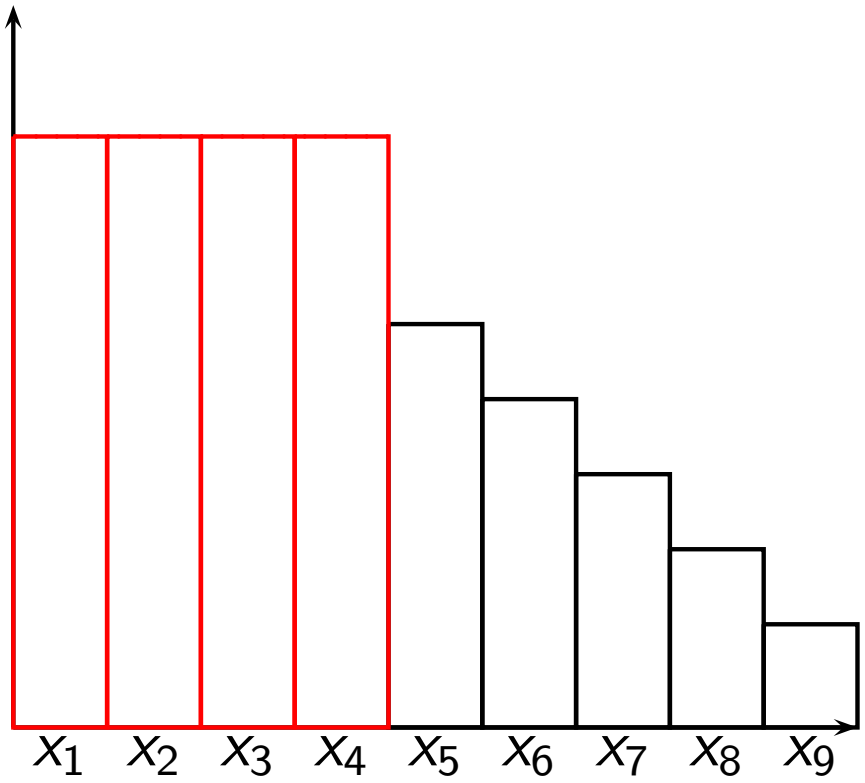

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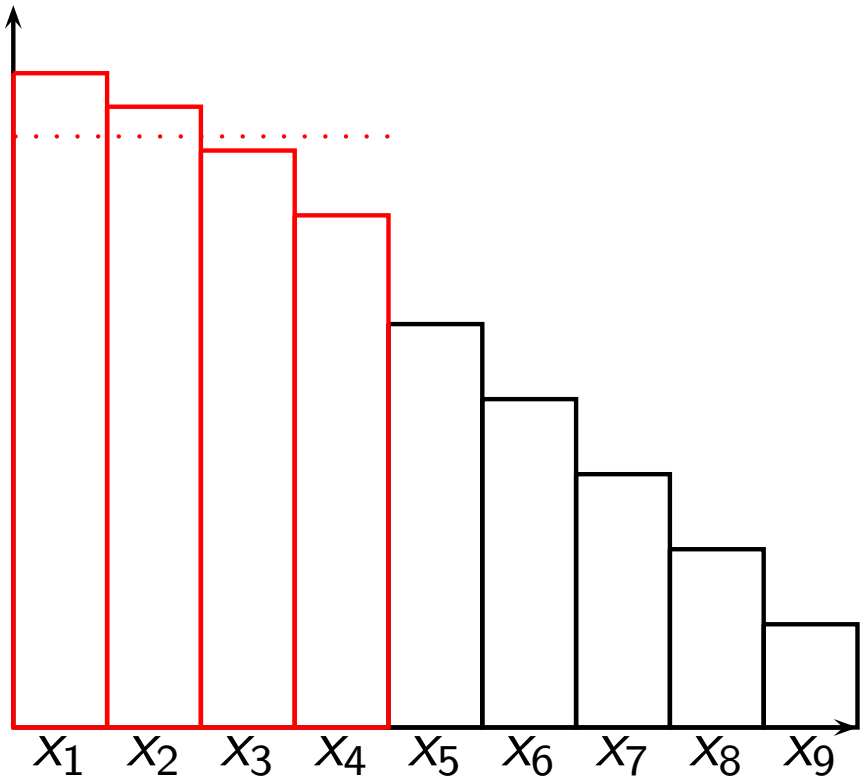

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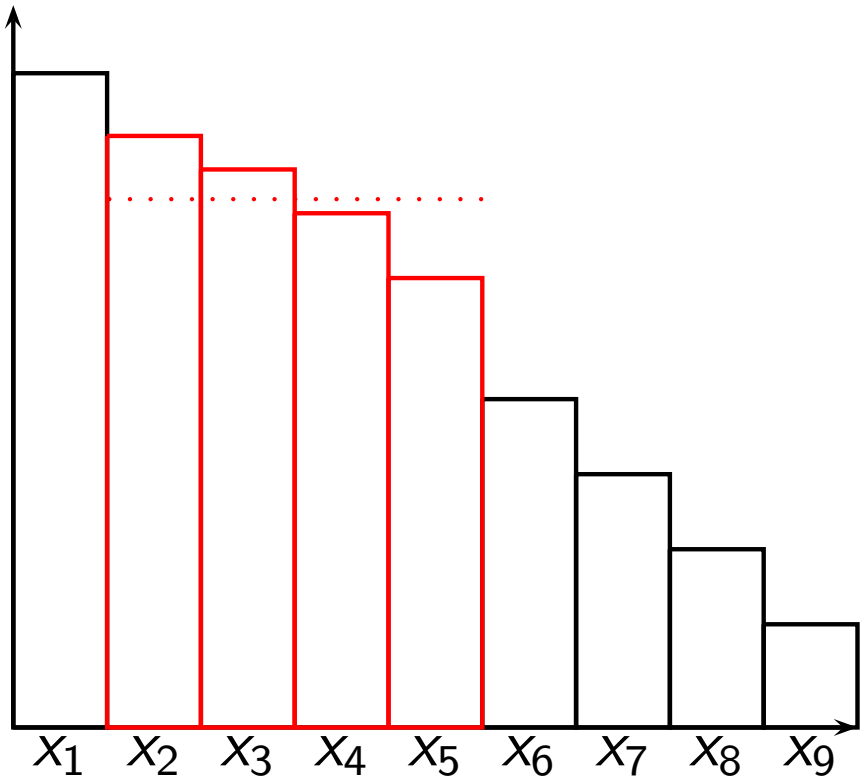

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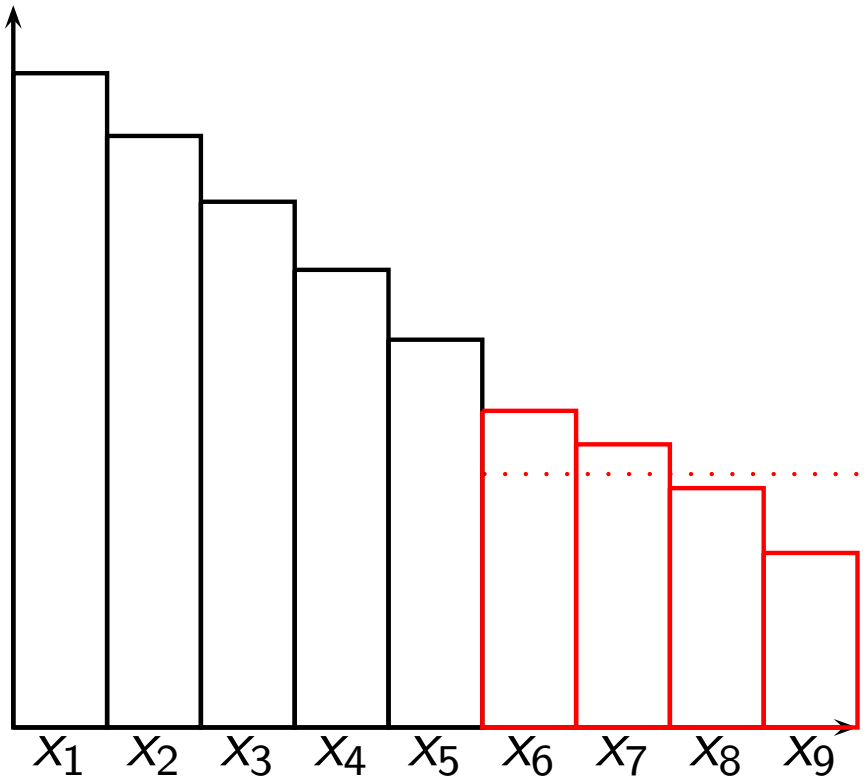

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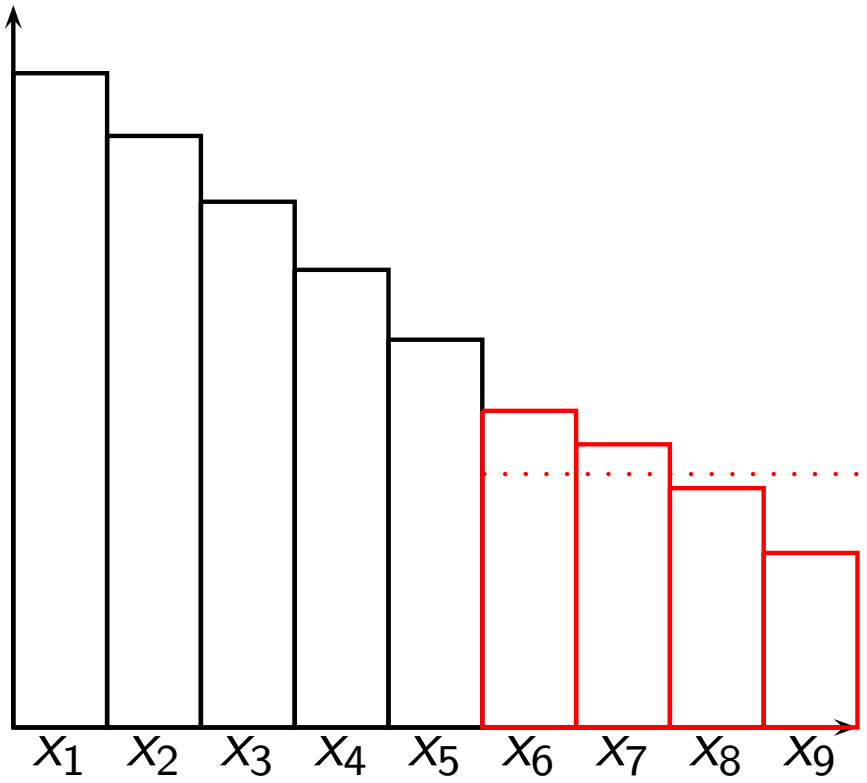

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Discrete-time affine dynamical system, for one loop iteration

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- Speed of convergence $\Rightarrow$ eigenvalues of $A^T A$
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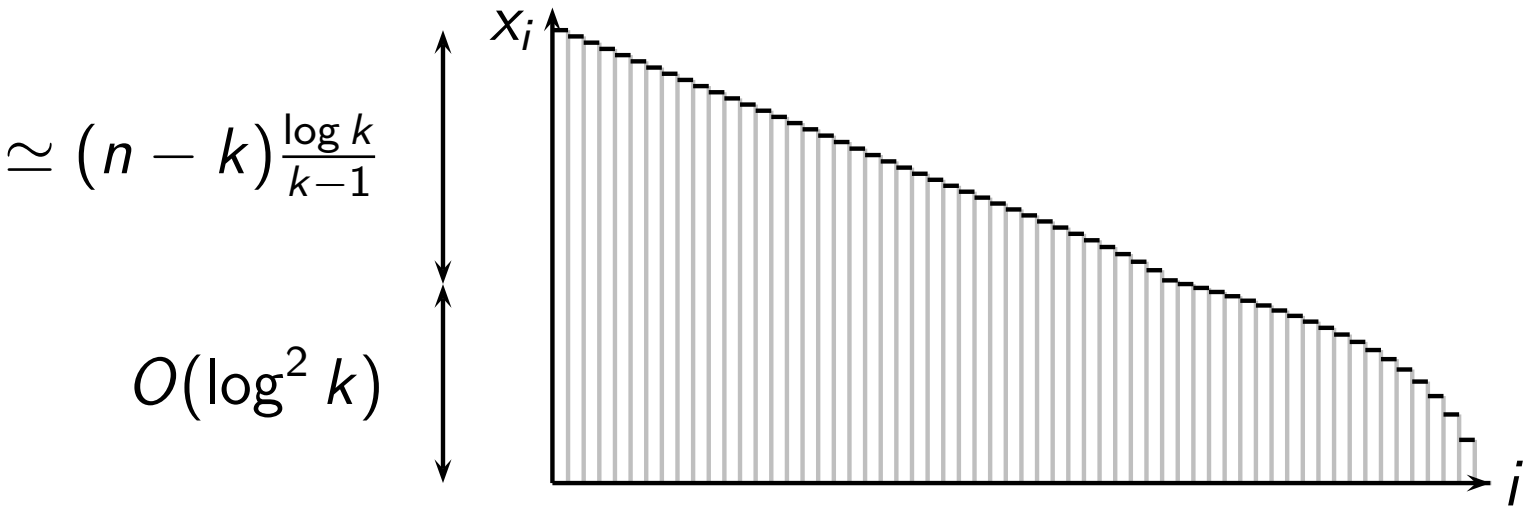

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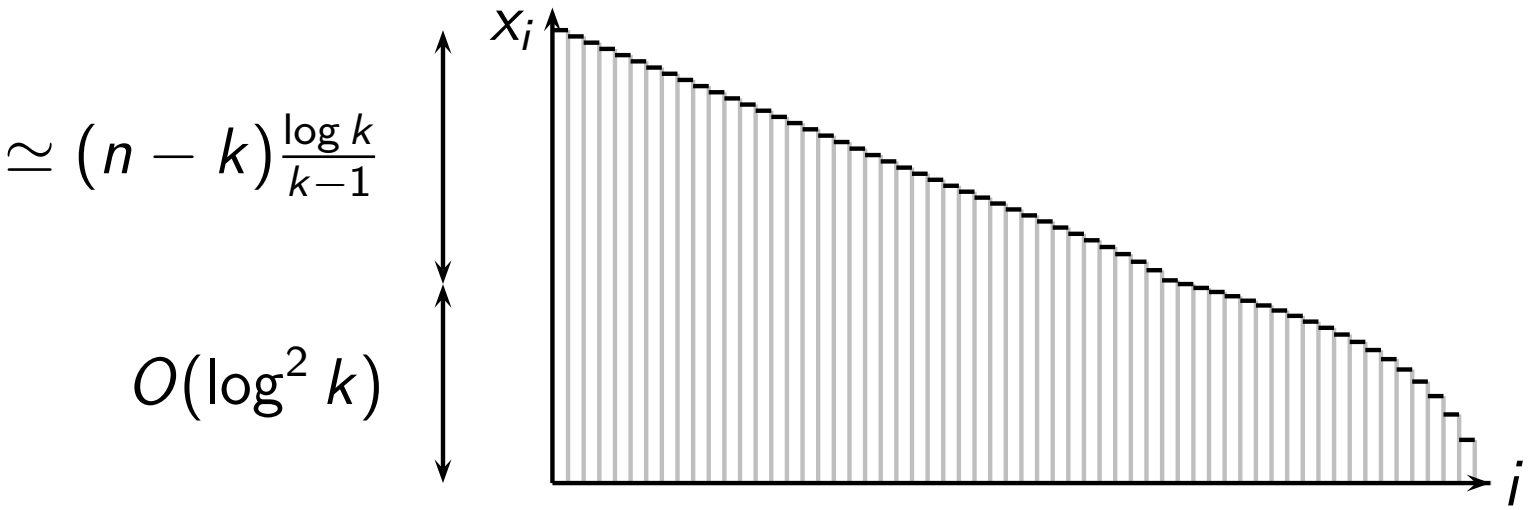

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One can solve $X = AX + \Gamma$, find the eigenvalues of $A^T A$, and remove the regularity assumption... cumbersome...

Neumaier's reducedness parameter

$$\nu := \max_{i \leq n-k} \frac{1}{n-i} \left(\frac{\sum_{j \leq i} x_j}{i} - \frac{\sum_{j \leq n} x_j}{n} \right).$$

- $(\sum_{j \leq i} x_j)/i$ is a smoothed proxy for x_i.
- Taking $i = 1$ gives $\|\mathbf{b}_1\| \leq \exp(\nu)^{n-1} \cdot (\det B)^{1/n}$.
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SIS [Ajt96]

The Small Integer Solution Problem

Given a uniform $A \in \mathbb{Z}_q^{m \times n}$, find $\mathbf{x} \in \mathbb{Z}^m$ s.t.:

$$0 < \|\mathbf{x}\| \leq \beta \quad \text{and} \quad \mathbf{x}^t \cdot A = \mathbf{0} \mod q.$$

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And many more.

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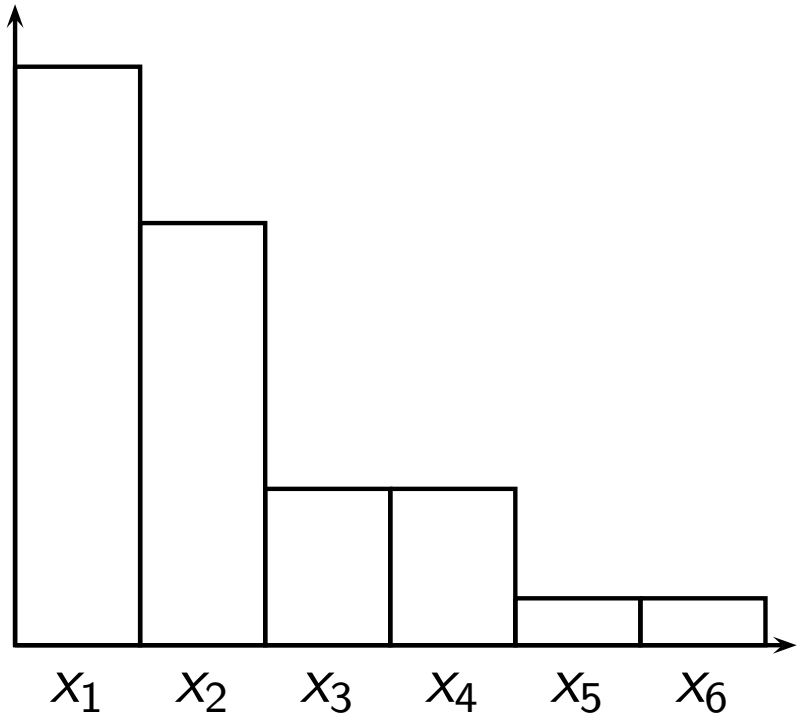

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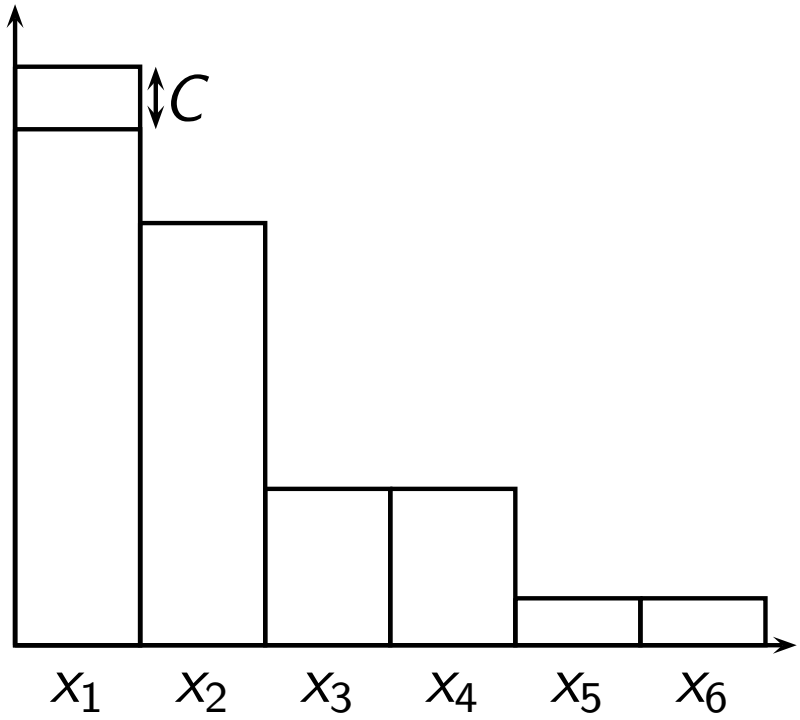

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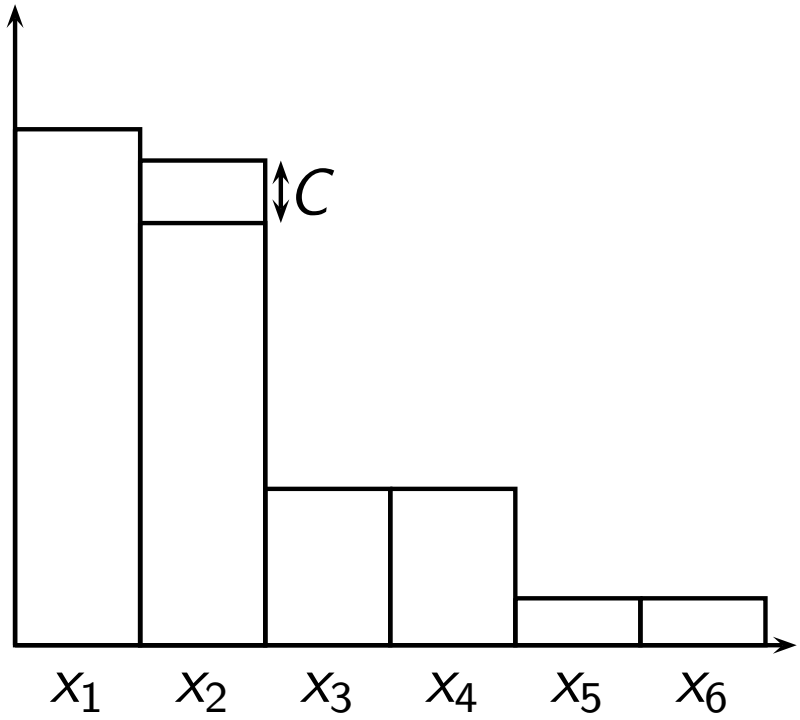

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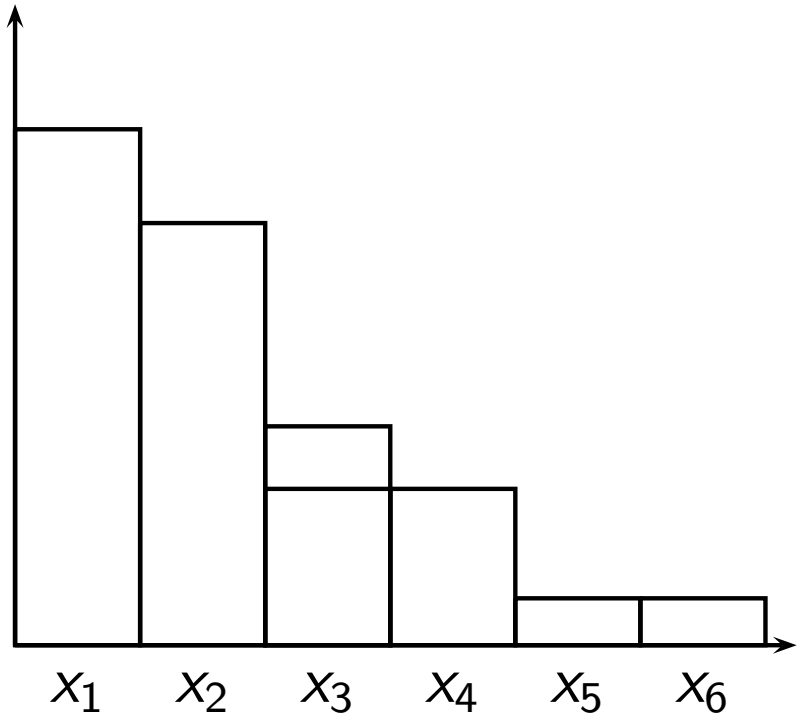

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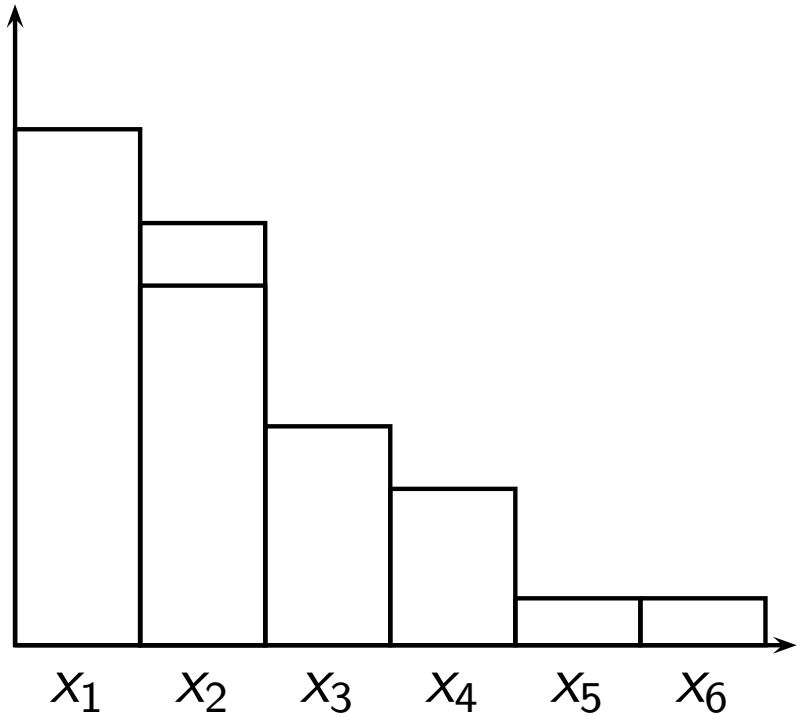

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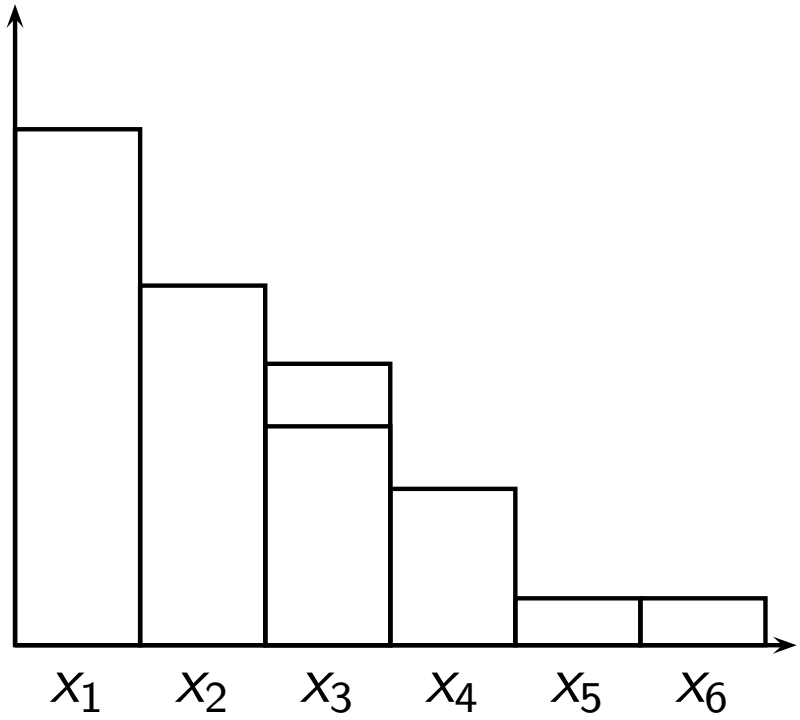

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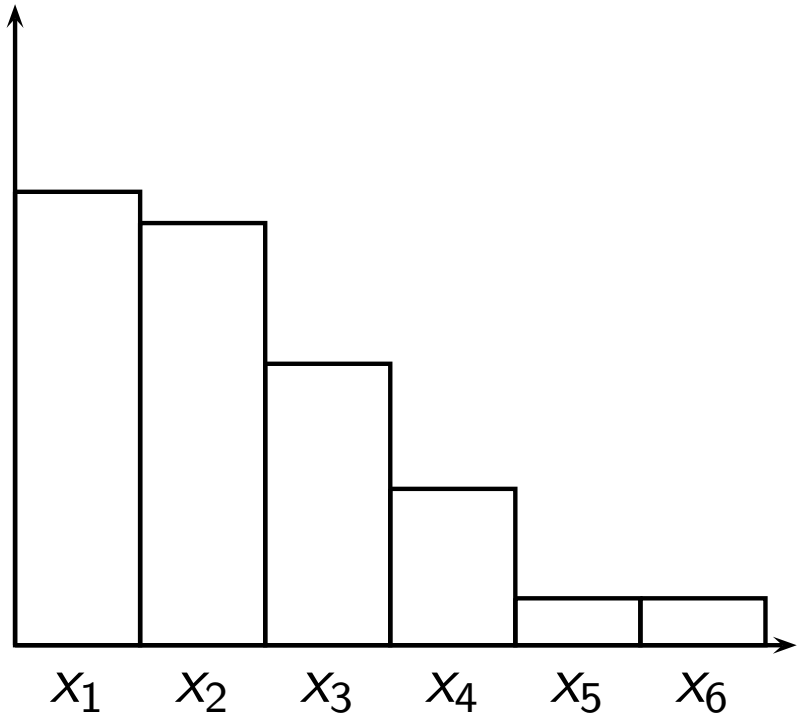

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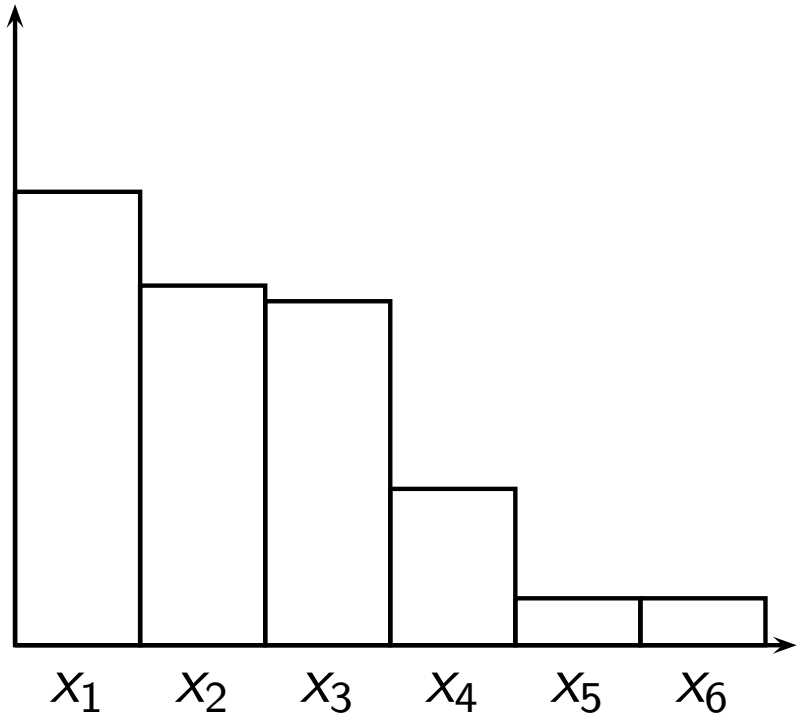

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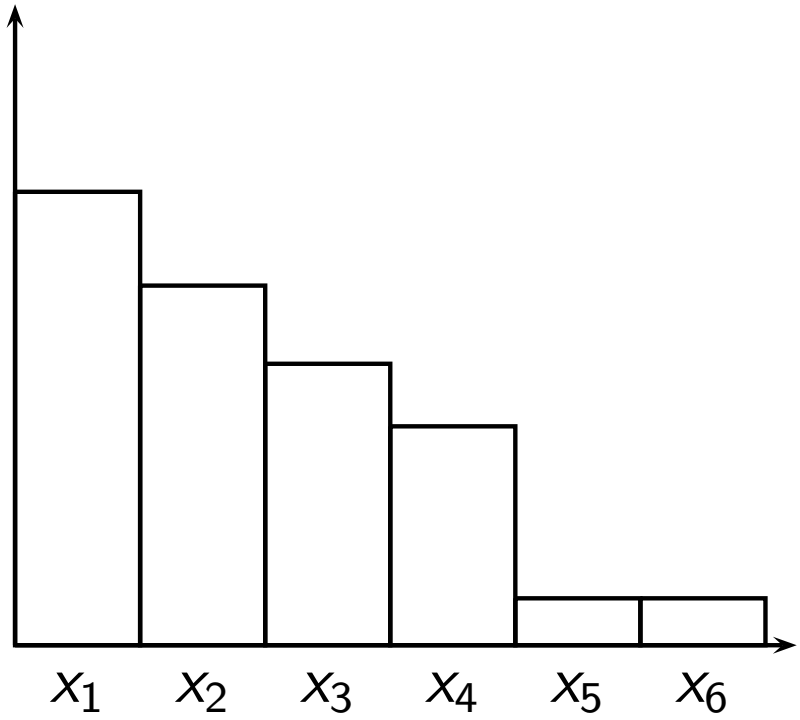

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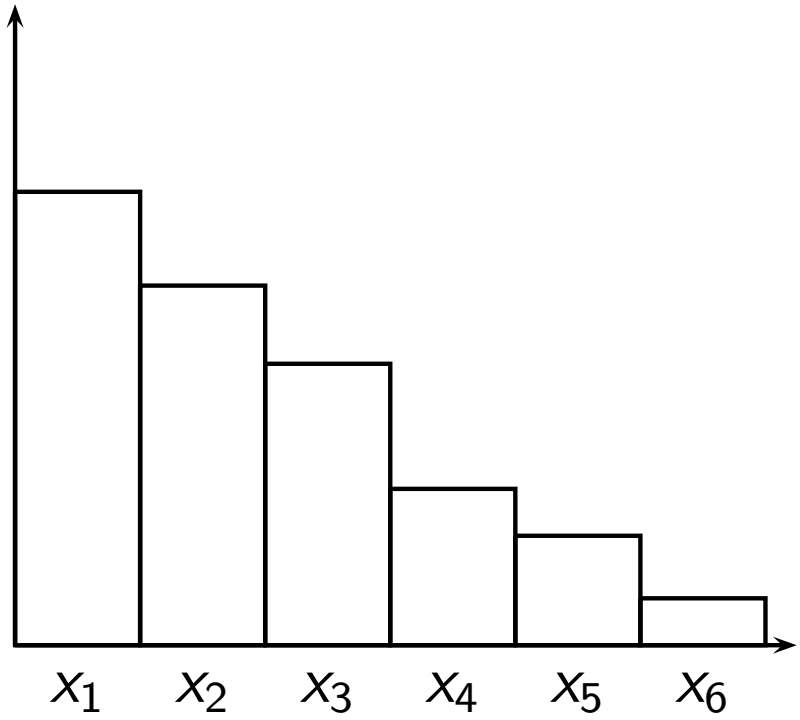

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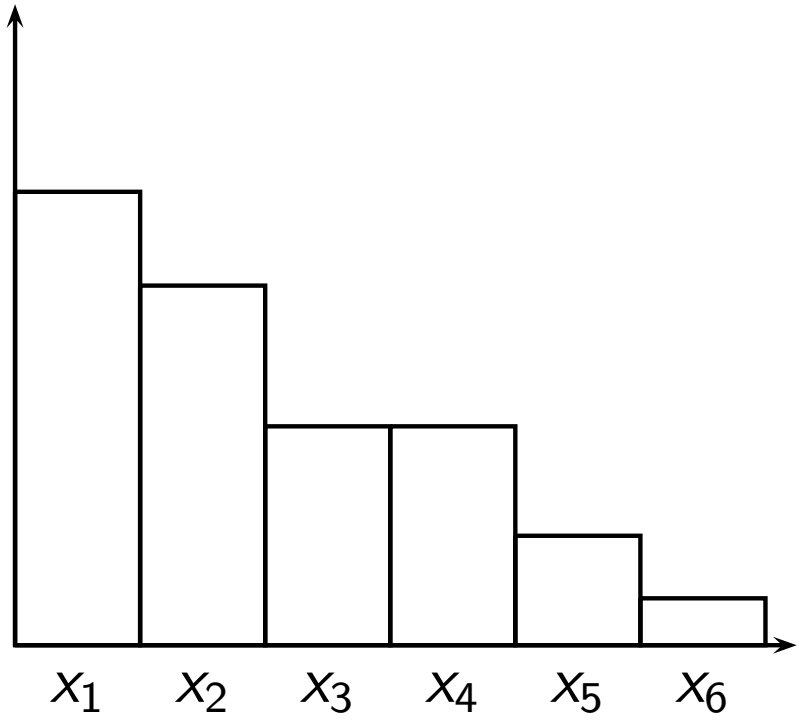

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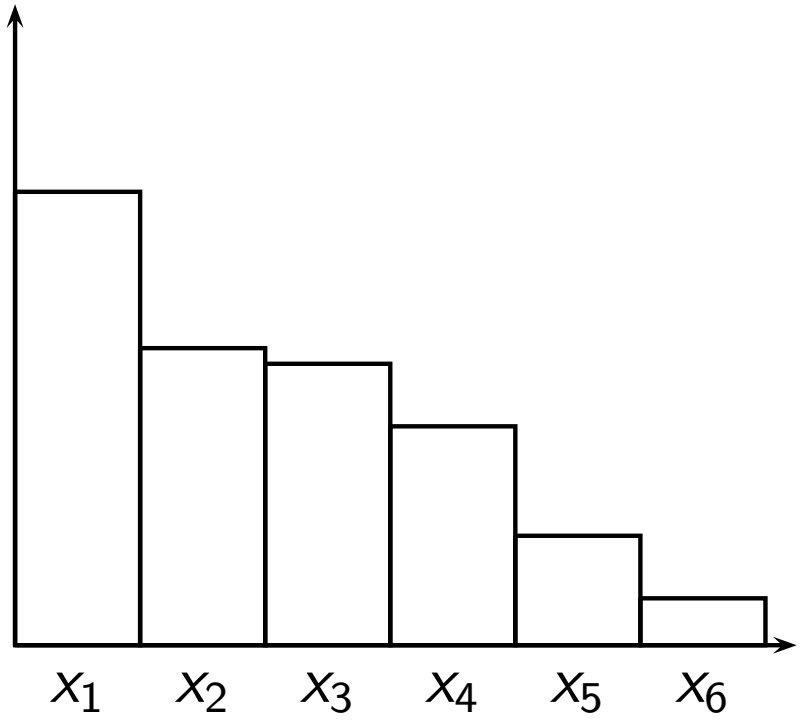

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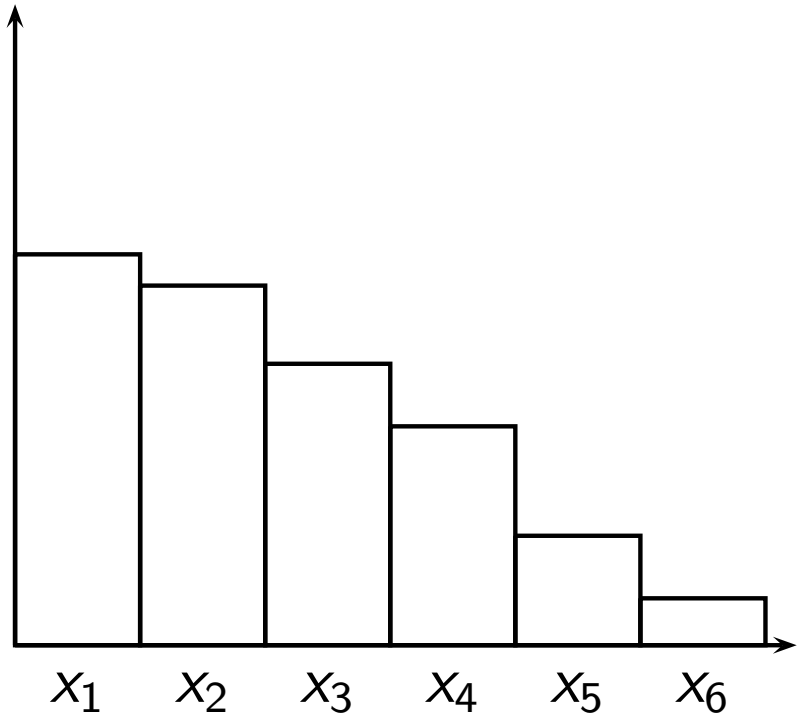

Convergence of LLL

The LLL potential

$$\Pi := \prod_{i \leq n} (n - i + 1) \cdot x_i.$$

- It is the weighted amount of sand to be moved to the right.
- For each swap, it decreases by at least a constant.

Number of loop iterations of LLL

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Text-book LLL:

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- GSO rationals have bit-lengths $O(n \log \|B\|)$

$\Rightarrow$ Cost is $\widetilde{O}(n^5 \log^2 \|B\|)$

Improvements:

- Use floating-point GSO [NgSt05]:
$$\widetilde{O}(n^4 \log^2 \|B\|)$$
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Faster LLL-type reduction [NeSt16]

Ideas:

- Use a BKZ-like **global** strategy
- In the k-dimensional blocks, make a recursive call
- Make the blocks overlap by half only
- At the bottom of the recursion, use a quasi-linear 2-dimensional algorithm.

This is solving a local-global dilemma:

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Roadmap

1. Background on lattices
2. The lattice reduction framework
3. Strong but slow: BKZ
4. Solving the SIS problem
5. Weak but fast: LLL

What to remember from this talk?

- Lattice reduction is used to solve the approximate variants of SVP/uSVP/HSVP/SIVP/...
- The process is driven by the r_{ii}'s
- Time $2^k \Leftrightarrow$ approx. factor $\gamma = k^{O(n/k)}$
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2. Understand the relationship between global and local flattening
3. Faster LLL-type reduction: $\widetilde{O}(n^\omega \log \|B\|)$?
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